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Universal $\mathcal{R}$-matrix of reductive Lie algebras and quantum integrable systems from its colour representation

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Abstract. The universal $\mathcal{R}$-matrix intertwining between the coproduct structures related to the universal deformations of reductive Lie algebras has been found in an explicit form using twisting. A possible colour representation of this $\mathcal{R}$-matrix and its application to the integrable systems are shown using the example of a deformed $gl(n)$ algebra.

Introduction

The universal $\mathcal{R}$-matrix is an abstraction of the quantum R-matrix used in the theory of integrable models. In the study of Hopf algebras it becomes an object of its own interest and plays a central role in the definition of quasitriangularity [1]. The explicit construction of the universal $\mathcal{R}$-matrix is a difficult task in general, but following the prescription given by Drinfeld [2] such an object has been found for deformations of semisimple [3–5] as well as Affine [6] Lie algebras. In [6, 7] the uniqueness of such solutions within the given ansatz has also been shown. In the recent past a multiparameter deformation of all reductive Lie algebras has been formulated [8] with the property of universality in a certain class. Our first goal is to build a universal $\mathcal{R}$-matrix for this deformation and show that it is indeed a quasitriangular Hopf algebra. In order to do so we exploit the twisting method [9] for introducing new parameters as well as for making the transition to the reductive case. The physical motivation behind this construction is to go back to the theory of integrable models again and use such an $\mathcal{R}$-matrix for building the associated quantum R-matrix and Lax operators with spectral as well as colour parameters. The colour parameters are provided by the eigenvalues of the central generators of the reductive Lie algebra in a given representation, whereas the spectral parameters may be introduced using a suitable Yang–Baxterization scheme. We underline the full generality of our construction. Beside its applicability, in principle, to simple Lie algebras of any type and to their semisimple and reductive generalizations, it includes through the representations of the universal $\mathcal{R}$-matrix an infinite variety of cases. In fact, in the application to the theory of quantum integrable systems we may consider different finite-dimensional representations of the $\mathcal{R}$-matrix yielding the corresponding braid-group representations (BGR), coloured BGR along with the related $L^{\pm}$ matrices used in the Faddeev–Reshetikhin–Takhtajan algebra (FRT) [10]. Moreover, whenever it is possible to formulate a suitable Yang–Baxterization scheme for the FRT algebra we are able to construct genuine spectral-parameter-dependent Lax operators and quantum

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R-matrices, thus generating new classes of quantum integrable models, beside a generalization of the known Toda field models on lattice. This project has been successfully carried through for $gl(n+1)$ and the result is reported in this paper, which is organized as follows. In section 1 we construct the universal $\mathcal{R}$-matrix for the deformations of all reductive Lie algebras, based on the Hopf algebra introduced in [8] which is universal within a certain class; the result of section 1 is to show that such a Hopf algebra is (pseudo [1] or essentially [11]) quasitriangular. Starting from section 2 we specialize to the case $gl(n+1)$. In section 2 we construct the coloured braid-group representations (CBGR) and coloured Faddeev–Reshetikhin–Takhtajan algebra (CFRT). In section 3 we introduce spectral parameters using a Yang–Baxterization scheme and build spectral-parameter-dependent quantum R-matrices and corresponding Lax operators. The applications to integrable models are shown in section 4, where particular attention in paid to the generalization of the well known quantum Toda field model on lattice. Section 5 is the concluding section. An appendix shows how the explicit form of the universal $\mathcal{R}$-matrix changes for a different ordering.

1. Construction of the universal R-matrix

Let g be a reductive Lie algebra of rank N. Namely g is the direct sum of, say, M simple Lie algebras plus an Abelian centre. $H_1, H_2, \ldots, H_N$ is the basis of the Cartan algebra of which $H_i (1 \leqslant i \leqslant N_1)$ span the semisimple part and the remaining $N - N_1$ number of $H_\alpha, (N_1 < \alpha \leqslant N)$ belong to the centre of the algebra. Let $\mathbf{a}_k$ be vectors with N components $a_{\ell k}$, such that $a_{\alpha k} = 0$ for all k and $(N_1 < \alpha \leqslant N)$, while $a_{ij} = 2(\alpha_i \cdot \alpha_j)/(\alpha_i \cdot \alpha_i)$ with $(1 \leqslant i, j \leqslant N_1)$ are the entries of the Cartan matrix related to the semisimple part. Let $X_i^\pm$ be generators associated to the simple roots α_i. It is established that [8] for such algebras the universal deformation $\mathcal{U}_q(g)$ can be defined so that each simple component remains the same as that of the standard one-parameter quantization with relations

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q_{ii}^{\frac{H_i}{2}} - q_{ii}^{-\frac{H_i}{2}}}{q_{ii}^{\frac{1}{2}} - q_{ii}^{-\frac{1}{2}}} \equiv \delta_{ij} [H_i]_{q_{ii}} \tag{1.1}$$

and $f(\mathbf{H}) X_i^\pm = X_i^\pm f(\mathbf{H} \pm \mathbf{a}_i)$ plus the Serré relations

$$\sum_{0 \leqslant k \leqslant n} (-1)^k \binom{n}{k}_{q_{ii}} (X_i^\pm)^{n-k} X_j^\pm (X_i^\pm)^k = 0 \qquad i \neq j \tag{1.2}$$

with $n = 1 - a_{ij}$ and the notation $q_{ii} = \mathrm{e}^{h_{\rho(i)}(\alpha_i \cdot \alpha_i)}$, where $(1 \leqslant \rho(i) \leqslant M)$ counts the number of simple components. Since ρ is constant on each simple component, we may also use the notation $q_\rho = \mathrm{e}^{h_{\rho(i)}}$. The remaining deformation parameters on the other hand, can be relegated to the coalgebra structure defining the corresponding coproducts as

$$\Delta(X_i^\pm) = X_i^\pm \otimes \Lambda_i^\pm + (\Lambda_i^\pm)^{-1} \otimes X_i^\pm \tag{1.3}$$

$$\Delta(H_i) = H_i \otimes \mathbf{I} + \mathbf{I} \otimes H_i \tag{1.4}$$

where $\Lambda_i^\pm$, containing the parameters $v_{i\alpha}, t_{ik}$ $(t_{ik} = -t_{ki})$, has the form

$$\Lambda_i^\pm \equiv q_{ii}^{\frac{H_i}{4}} \mathrm{e}^{\pm\frac{1}{2}(\sum_{j,k} t_{ik} a^{kj} H_j + \sum_\alpha v_{i\alpha} H_\alpha)}. \tag{1.5}$$

The antipode (S) and co-unit (ϵ) are given by

$$S(X_i^{\pm}) = -q_{ii}^{\pm\frac{1}{2}} X_i^{\pm} \qquad S(H_i) = -H_i \qquad \text{and} \qquad \epsilon(X_i^{\pm}) = \epsilon(H_i) = 0\,. \tag{1.6}$$

This defines the universal deformation as a Hopf algebra.

Our aim here is to find a universal $\mathcal{R}$-matrix ($\mathcal{R}$ is in a completion of $\mathcal{U}_q(g)\otimes\mathcal{U}_q(g)$ [1]) corresponding to the above Hopf algebra, which would intertwine between the coproduct Δ (equations (1.3)–(1.5)) and its permuted form $\tilde{\Delta} = \sigma\Delta$ as

$$\mathcal{R}\Delta(a) = \tilde{\Delta}(a)\mathcal{R} \qquad \forall a \in \mathcal{U}_q(g) \tag{1.7}$$

and satisfy the properties

$$(\Delta\otimes\mathbf{I})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (\mathbf{I}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}\,. \tag{1.8}$$

Such an $\mathcal{R}$ would naturally endow the above Hopf algebra with quasitriangularity.

Following the argument of [3–6] we first construct the universal $\mathcal{R}$-matrix for the case $t_{ik} = v_{i\alpha} = 0$, i.e. for the semisimple but untwisted deformed Lie algebra only depending on the deformation parameters q_ρ ($1 \leqslant \rho \leqslant M$). Denoting it by $\mathcal{R}_0$, we have

$$\mathcal{R}_0 = \check{R}K \tag{1.9}$$

where K is expressed in terms of the Cartan generators only:

$$K = \exp\left(\sum_{ij} h_{\rho(i)} \frac{(\alpha_i\cdot\alpha_i)}{2}\frac{(\alpha_j\cdot\alpha_j)}{2} d^{ij} H_i\otimes H_j\right) \tag{1.10}$$

with $d^{ij} = (d^{-1})_{ij}$, where $d_{ij} = (\alpha_i\cdot\alpha_j)$ is the symmetrized Cartan matrix. $\check{R}$ is then given in a factorized form as

$$\check{R} = \prod_{\rho=1}^{M}(\check{R}^{(\rho)}) \qquad \text{with} \qquad \check{R}^{(\rho)} = \prod_{\gamma\in\Delta_+^\rho}(\check{R}_\gamma^{(\rho)}) \tag{1.11}$$

where Δ_+^ρ is the set of *all* positive roots belonging to the ρth simple component with the prescribed normal ordering [6]. We have, in turn,

$$\check{R}_\gamma^{(\rho)} = \exp_{q_{\gamma\gamma}^{-1}}\left(a_\gamma^{-1}(q_\rho - q_\rho^{-1})(e_\gamma\otimes e_{-\gamma})\right) \tag{1.12}$$

where $\exp_q$ is the q-exponential function $\exp_q(x) = \sum_{n\geqslant 0}\frac{x^n}{(n)_q!}$ with $(n)_q := \frac{q^n-1}{q-1}$ and a_γ is defined by the commutation relation among the following elements of the *q-deformed* Cartan–Weyl basis [6]:

$$[e_\gamma, e_{-\gamma}] = a_\gamma \frac{q_{\gamma\gamma}^{h_\gamma/2} - q_{\gamma\gamma}^{-h_\gamma/2}}{q_\rho - q_\rho^{-1}}\,. \tag{1.13a}$$

In the above expression (1.13a), if the index γ corresponds to the non-simple root $\alpha_\gamma = \sum\alpha_i$ for certain simple roots α_i, then $h_\gamma = \sum h_i$ and

$$h_i = \frac{(\alpha_i\cdot\alpha_i)}{2}H_i \tag{1.13b}$$

$$e_{\pm i} = \left[\frac{(\alpha_i\cdot\alpha_i)}{2}\right]_{q_\rho^2}^{1/2} q_{ii}^{-1/4} X_i^{\pm} q_{ii}^{\mp H_i/4}\,. \tag{1.13c}$$

The $\mathcal{R}_0(q_{ii})$-matrix thus obtained complies with the coproduct of the form (1.3) and (1.4), where the $\Lambda_i^{\pm}$ operators contain only q_{ii} as deforming parameters:

$$\mathcal{R}_0\Delta_0(a) = \tilde{\Delta}_0(a)\mathcal{R}_0 \qquad a \in \mathcal{U}_q(g) \tag{1.14}$$

where

$$\Delta_0(X_i^{\pm}) = X_i^{\pm} \otimes q_{ii}^{\frac{H_i}{4}} + q_{ii}^{-\frac{H_i}{4}} \otimes X_i^{\pm}. \tag{1.15}$$

Next we include the remaining parameters t_{ij} and $v_{i\alpha}$ to obtain $\mathcal{R}(q_{ii}, t_{ij}, v_{i\alpha})$ by implementing the twisting transformation [9] and suitably choosing the twisting operators for the reductive case.

We thus obtain the following universal $\mathcal{R}$-matrix associated to the universal deformation of any reductive Lie algebra:

$$\mathcal{R}(q_{ii}, t_{ij}, v_{i\alpha}) = \mathcal{G}^{-1}(v_{i\alpha})\mathcal{F}^{-1}(t_{ij})\mathcal{R}_0(q_{ii})\mathcal{F}^{-1}(t_{ij})\mathcal{G}^{-1}(v_{i\alpha}) \tag{1.16}$$

where $\mathcal{R}_0$ is given in explicit form through (1.9)–(1.12) and the twisting operator $\tilde{\mathcal{G}}$ is given by

$$\begin{aligned}\tilde{\mathcal{G}}(t_{ij}, v_{i\alpha}) &\equiv \mathcal{G}(v_{i\alpha})\mathcal{F}(t_{ij}) \\ &= \exp\left\{\tfrac{1}{2}\left(\sum_{j\alpha} v_{j\alpha}a^{jk}(H_k \otimes H_\alpha - H_\alpha \otimes H_k) + \sum_{ijkr} a^{ki} t_{kr} a^{rj} H_i \otimes H_j\right)\right\} \\ &\quad (a^{ij} = (a^{-1})_{ij}).\end{aligned} \tag{1.17}$$

Note that the property $\tilde{\mathcal{G}}_{21}^{-1} = \tilde{\mathcal{G}}_{12}$, which is essential for constructing a consistent $\mathcal{R}$-matrix through (1.16), is satisfied due to the antisymmetry of t_{kr} and irrespective of the fact that $v_{j\alpha}$ is not antisymmetric (recall that $\{v_{j\alpha}\}$ may have $(N - N_1) \times N_1$ number of independent elements). For showing that the operator (1.17) may be taken as a twisting operator, following Reshetikhin [9] $\tilde{\mathcal{G}}$ has to also satisfy

$$(\Delta \otimes \mathbf{I})\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_{13}\tilde{\mathcal{G}}_{23} \qquad (\mathbf{I} \otimes \Delta)\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_{13}\tilde{\mathcal{G}}_{12} \tag{1.18}$$

along with

$$\tilde{\mathcal{G}}_{12}\tilde{\mathcal{G}}_{13}\tilde{\mathcal{G}}_{23} = \tilde{\mathcal{G}}_{23}\tilde{\mathcal{G}}_{13}\tilde{\mathcal{G}}_{12} \tag{1.19}$$

which, however, can be checked easily.

We may see now that the coproduct is changed under such twisting as

$$\Delta(a) = \tilde{\mathcal{G}}\Delta_0(a)\tilde{\mathcal{G}}^{-1} \qquad a \in \mathcal{U}_q(g) \tag{1.20}$$

yielding, in particular,

$$\begin{aligned}\Delta(X_i^{\pm}) &= \tilde{\mathcal{G}}\Delta_0(X_i^{\pm})\tilde{\mathcal{G}}^{-1} \\ &= X_i^{\pm} \otimes q_{ii}^{\frac{H_i}{4}} e^{\pm\frac{1}{2}(\sum_{j,k} t_{ik}a^{kj}H_j + \sum_\alpha v_{i\alpha}H_\alpha)} + q_{ii}^{-\frac{H_i}{4}} e^{\mp\frac{1}{2}(\sum_{j,k} t_{ik}a^{kj}H_j + \sum_\alpha v_{i\alpha}H_\alpha)} \otimes X_i^{\pm}\end{aligned} \tag{1.21}$$

and $\Delta(H_i) = \Delta_0(H_i)$ and this coincides exactly with (1.3)–(1.5) with all non-trivial parameters q_{ii}, t_{ij} and $v_{i\alpha}$.

Consequently, the universal $\mathcal{R}$-matrix thus obtained has the properties (1.7) and (1.8) and this concludes the proof that the algebra introduced in [8] is (pseudo [1]) quasitriangular. The total number of independent parameters contained in $\mathcal{R}$ is $M+\frac{1}{2}N_1(N_1-1)+N_1(N-N_1)$ for a reductive Lie algebra of rank N with M simple components plus a $(N-N_1)$-dimensional centre.

It may be worthwhile to note here that the explicit form of the universal $\mathcal{R}$-matrix can be given in a slightly different form by inverting the ordering of K and $\check{R}$ in (1.9). This yields

$$\mathcal{R} = \check{R}K = K\check{R}' \tag{1.22a}$$

where

$$\check{R}' = \prod_{\rho=1}^{M} \prod_{\gamma\in\Delta_+^\rho} \exp_{q_{\gamma\gamma}^{-1}}(a_\gamma^{-1}(q_\rho - q_\rho^{-1})(\tilde{e}_\gamma \otimes \tilde{e}_{-\gamma})) \tag{1.22b}$$

with

$$\tilde{e}_\gamma = e_\gamma\, q_\rho^{h_\gamma} \qquad \text{and} \qquad \tilde{e}_{-\gamma} = q_\rho^{-h_\gamma} e_{-\gamma}\,. \tag{1.23}$$

The proof of this result is given in the appendix.

It is worth noticing that the algebraic relations remain unchanged under the rescaling (1.23) of the generators. This fact should be remembered in considering representations of the $\mathcal{R}$-matrix, since representation theory only involves the algebraic part of the Hopf structure. The rescaling becomes trivial when representing the $\mathcal{R}$-matrix as a finite-dimensional matrix, hence the relative order of K and $\check{R}$ is such cases becomes irrelevant.

2. Colour representations

A universal $\mathcal{R}$-matrix $\in U_q(g) \otimes U_q(g)$ (to be rigorous we should say that $\mathcal{R}$ is in a completion of $U_q(g) \otimes U_q(g)$) related to the universal deformation of any reductive Lie algebra has been constructed in the previous section. Specializing this general form (1.16) to a particular Lie algebra, which will be reflected in the corresponding choice of root systems, form of the Cartan matrix a_{ij} and normal ordering in Δ_+, one can derive the universal $\mathcal{R}$-matrix related to Lie algebras of A, B, C, D, G etc types together with their semisimple as well as reductive generalizations. The finite-dimensional representations of these $\mathcal{R}$-matrices yields the corresponding braid-group representations (BGR), colour BGR along with the related $L^{\pm}$ matrices used in the Faddeev–Reshetikhin–Takhtajan algebra [10]. A suitable Yang–Baxterization scheme for the FRT algebra related to these Lie algebras, allows us to construct genuine spectral-parameter-dependent Lax operators and quantum R-matrices generating thus new classes of quantum integrable models.

We would like to demonstrate this promising scheme using the example of the multiparameter deformed reductive $U_q(gl(n+1))$ algebra. It is straightforward to see that in this case the Cartan matrix

$$a_{ij} = 2\delta_{ij} - \delta_{ij+1} - \delta_{ij-1}$$

would correspond to $sl(n+1)$ with $H_i, (i=1,\dots,n)$ constituting the Cartan subalgebra and $X_i^{\pm}, (i=1,\dots,n)$ being generators corresponding to simple roots. Moreover, in this case we have $M=1$ giving $q_{ii}=\mathrm{e}^{h_{(\wp)}(\alpha_i\cdot\alpha_i)}=\mathrm{e}^{2h_{(\wp)}}\equiv q^2$ and also $N_1=n$, $N=n+1$. Therefore for such a deformed algebra we have a single central element $H_\alpha=\mathbf{Z}$ and the related deforming parameters reduce to only $v_{\alpha j}=v_j$, $(j=1,\dots,n)$.

As a result we get the universal $\mathcal{R}$-matrix for $U_q(gl(n+1))$ as

$$\mathcal{R}^+(q,\phi_{ij},v_j)=\mathcal{G}^{-1}(v_j)\mathcal{F}^{-1}(\phi_{ij})K(q)\breve{R}'^+(q)\mathcal{F}^{-1}(\phi_{ij})\mathcal{G}^{-1}(v_j) \tag{2.1}$$

where

$$\breve{R}'^+=\prod_{\gamma\in\Delta_+}\exp_{q_{\gamma\gamma}^{-1}}\left(a_\gamma^{-1}\Lambda(\tilde{e}_\gamma\otimes\tilde{e}_{-\gamma})\right) \tag{2.2}$$

where $\Lambda=q-q^{-1}$, a_γ as defined in [6] and $\tilde{e}_{\pm\gamma}$ correspond to the roots $\gamma=\sum_{s=i}^{j}\alpha_s$ (for $i<j$ and α_s simple) which will be denoted below as $\tilde{e}_{ij}$. Note that we use the generators $\tilde{e}_{\pm\gamma}$ as in (1.23), since we have adopted here the ordering (1.22*a*) for defining our $\mathcal{R}^+$-matrix.

For the other operators in (2.1) likewise we get

$$K(q)=q^{\sum_{ij}a^{ij}H_i\otimes H_j} \tag{2.3}$$

$$\mathcal{F}(\phi_{ij})=\exp\Bigg(\sum_{ij}\phi_{ij}H_i\otimes H_j\Bigg) \tag{2.4}$$

where $\phi_{ij}\equiv\frac{1}{2}\sum_{kr}a^{ki}t_{kr}a^{rj}$ and

$$\mathcal{G}(v_j)=\exp\Bigg(-\tfrac{1}{2}\sum_{jk}v_ja^{jk}(\mathbf{Z}\otimes H_k-H_k\otimes\mathbf{Z})\Bigg). \tag{2.5}$$

Similarly one may obtain also another $\mathcal{R}$-matrix solution as

$$\begin{aligned}\mathcal{R}^-(q,\phi_{ij},v_j)&=\sigma(\mathcal{R}^+)^{-1}\\&=\mathcal{G}^{-1}(v_j)\mathcal{F}^{-1}(\phi_{ij})\breve{R}'^-(q)K^{-1}(q)\mathcal{F}^{-1}(\phi_{ij})\mathcal{G}^{-1}(v_j)\end{aligned} \tag{2.6}$$

where $\breve{R}'^-$ is the reduced matrix with reverse ordering:

$$\breve{R}'^-=\prod_{\gamma\in\Delta_+}\exp_{q_{\gamma\gamma}^{-1}}\left(-a_\gamma^{-1}\Lambda(\tilde{e}_{-\gamma}\otimes\tilde{e}_\gamma)\right) \tag{2.7}$$

obtained from $(\breve{R}_{21}'^+)^{-1}$ and due to $\mathcal{F}_{21}^{-1}=\mathcal{F}_{12}$, $\mathcal{G}_{21}^{-1}=\mathcal{G}_{12}$ and ${K_{21}}^{-1}={K_{12}}^{-1}$ in the present case with symmetric a_{ij}.

It is interesting to obtain the universal $\mathcal{R}$-matrix related to $U_q(gl(2))$ from our general solution (2.1) and (2.2). In this particular case with $n=1$ the parameters ϕ_{ij} vanish giving $\mathcal{F}(\phi_{ij})=1$ and

$$\mathcal{R}(q,v_1)=\mathcal{G}^{-1}(v_1)\mathcal{R}_0\mathcal{G}^{-1}(v_1) \tag{2.1$'$}$$

Here

$$\mathcal{G}(v_1)=\exp\big(-\tfrac{1}{4}v_1(\mathbf{Z}\otimes H-H\otimes\mathbf{Z})\big)$$

and $\mathcal{R}_0$ represents the well known $U_q(sl(2,C))$ case [1,32]:

$$\begin{aligned}
\mathcal{R}_0 &= K\check{R}' \\
&= q_1^{\frac{H\otimes H}{4}} \exp_{q_1^{-1}}[(q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})(e_{12}q_1^{\frac{H}{2}} \otimes q_1^{-\frac{H}{2}} e_{21})] \\
&= q_1^{\frac{H\otimes H}{4}} \exp_{q_1^{-1}}[(q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})q_1^{-\frac{1}{2}}(q_1^{\frac{H}{4}} X^+ \otimes q_1^{-\frac{H}{4}} X^-)] \\
&= q_1^{\frac{H\otimes H}{4}} \sum_{n\geqslant 0} \frac{(1-q_1^{-1})^n q_1^{\frac{n(n-1)}{4}}}{[n]_{q_1}!} (q_1^{\frac{H}{4}} X^+)^n \otimes (q_1^{-\frac{H}{4}} X^-)^n .
\end{aligned}$$

Here $q_1 := q^{(\alpha\cdot\alpha)} = q^2$ and we have connected the standard notations $\exp_q(x) = \sum_{n\geqslant 0} \frac{x^n}{(n)_q!}$ with $(n)_q := \frac{q^n-1}{q-1}$ (see e.g. [6]) and $[n]_q = \frac{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$ (see e.g. [32]) using the identity

$$[n]_q! = (n)_{q^{-1}}!\, q^{\frac{n(n-1)}{4}} .$$

Such an $\mathcal{R}$-matrix for $U_q(gl(2))$ has also been constructed recently in [12]. Note that in the finite-dimensional fundamental representation π of $U_q(gl(n+1))$ the $\mathcal{R}^{\pm}$-matrices $\in U_q(gl(n+1)) \otimes U_q(gl(n+1))$ may be reduced to qualitatively different objects [13] yielding BGR $R^{\pm}$ or the $L^{\pm}$ matrices involved in the FRT algebra. In particular, one obtains

$$(\pi \otimes \mathbf{I})\mathcal{R}^{\pm}_{1\mathbf{2}} = L^{\pm}_{1\mathbf{2}} \tag{2.8}$$

$$(\pi \otimes \pi)\mathcal{R}^{\pm}_{\mathbf{12}} = R^{\pm}_{12} \tag{2.9}$$

where the bold face indices denote the infinite-dimensional space $U_q(g)$ and the ordinary indices its finite-dimensional fundamental representation. We intend to find in an explicit form the representation (2.8) and (2.9) for the universal $\mathcal{R}$-matrices (2.1) and (2.6) using the basis matrices E_{ij} with matrix elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for the fundamental representation of $gl(n+1)$ as

$$\pi(H_i) = E_{ii} - E_{i+1i+1} \tag{2.10}$$

$$\pi(X_i^+) = E_{ii+1} \qquad \pi(X_i^-) = E_{i+1i} \tag{2.11}$$

$$\pi(X_{ij}^+) = E_{ij+1} \qquad \pi(X_{ij}^-) = E_{j+1i} . \tag{2.12}$$

Using the matrix products $E_{ij}E_{kl} = \delta_{jk}E_{il}$ and the obvious relation

$$\pi(q_{ii}^{\pm H_i}) = 1 + E_{ii}(q_{ii}^{\pm 1} - 1) + E_{i+1i+1}(q_{ii}^{\mp 1} - 1)$$

it can easily be checked that the defining relation (1.13c) between the two sets of generators $\tilde{e}$ and X reduces for this particular representation simply to $\pi(\tilde{e}_{\pm i}) = q^{\mp 1}\pi(X_i^{\pm})$ yielding $\pi(\tilde{e}_{ij}) = E_{ij+1}q^{-(j-i+1)}$ ($j > i$) for the generator associated to the non-simple positive root $\gamma = \sum_{k=i}^{j} \alpha_k$ and $\pi(\tilde{e}_{ji}) = E_{j+1i}q^{(j-i+1)}$ for $-\gamma$. Therefore following [13], which is based on exploiting the above stated product rule of E_{ij} matrices and an identity involving a^{ij}, one finds from the universal $\mathcal{R}^{0+} = K\check{R}'^{+}$-matrix corresponding to $U_q(sl(n+1))$ without twisting:

$$\begin{aligned}
(\pi \otimes \mathbf{I})\mathcal{R}^{0+}_{1\mathbf{2}} &= L^{0+}_{1\mathbf{2}} \\
&= \sum_k \omega_k E_{kk} + \Lambda \sum_{ij} \omega_i (E_{ij+1}\hat{e}_{ji})
\end{aligned} \tag{2.13}$$

where we have defined $\omega_k = q^{\left(\sum_{s=k}^{n} H_s - \sum_{i=1}^{n} \frac{iH_i}{n+1}\right)}$ for $k = 1, \ldots, n$, while $\omega_{n+1} = q^{-\sum_{i=1}^{n} \frac{iH_i}{n+1}}$. Now invoking a similar representation $(\pi \otimes \mathbf{I})$ for the twisting operator $\mathcal{F}$ we get

$$(\pi \otimes \mathbf{I})\mathcal{F}_{12}^{-1} = F_{12}^{-1} = \exp\left(-\sum_{j=1}^{n}\sum_{k=1}^{n+1}(\phi_{kj} - \phi_{k-1j})E_{kk}H_j\right)$$
$$= \sum_{k=1}^{n+1} T_k E_{kk} \tag{2.14}$$

where

$$T_k = \mathrm{e}^{-\sum_{j=1}^{n}(\phi_{kj}-\phi_{k-1j})H_j}$$

by extending the range of parameters with trivial inclusion $\phi_{0j} = \phi_{n+1j} = 0$. However, it is remarkable that with respect to the reductive-type twisting operator $\mathcal{G}$ some non-trivial aspect may arise, since now we can consider $\mathbf{Z}$ to have different eigenvalues in different spaces, i.e. $(\pi \otimes \mathbf{I})\mathbf{Z} = \lambda I \otimes \mathbf{I}$, while $(\mathbf{I} \otimes \pi)\mathbf{Z} = \mu \mathbf{I} \otimes I$. Such pameters λ, μ may be taken as the colour parameters and the corresponding representations $(\pi_\lambda \otimes \mathbf{I}), (\mathbf{I} \otimes \pi_\mu)$ as the colour representations [14]. This gives an interesting possibility [15] of constructing colour FRT relations and the CBGR starting from the universal $\mathcal{R}$-matrix. Consequently from the general form of $\mathcal{R}^+$ (2.1) , using the representation (2.10)–(2.12), the colour representation $(\pi_\lambda \otimes \mathbf{I})\mathbf{Z} = \lambda I \otimes \mathbf{I}$ and the identity [13]

$$a^{jk} - a^{jk-1} = \begin{cases} 1 - \dfrac{j}{n+1} & k \leqslant j \\ -\dfrac{j}{n+1} & k > j \end{cases}$$

valid for $sl(n+1)$ we can derive after some algebra the expression

$$(\pi_\lambda \otimes \mathbf{I})\mathcal{G}_{12}^{-1} = G_{12}^{-1}(\lambda) = \sum_{k=1}^{n+1} W_k(\lambda) E_{kk} \tag{2.16}$$

where

$$\begin{aligned} W_k(\lambda) &= \mathrm{e}^{\frac{1}{2}\left(\lambda \sum_{jl}^{n} v_j a^{jl} H_l + \mathbf{Z}\left(\sum_{j=1}^{n} \frac{j v_j}{n+1} - \sum_{s=k}^{n} v_s\right)\right)} \qquad \text{for } k = 1, \ldots, n \\ W_{n+1}(\lambda) &= \mathrm{e}^{\frac{1}{2}\left(\lambda \sum_{jl}^{n} v_j a^{jl} H_l + \mathbf{Z} \sum_{j=1}^{n} \frac{j v_j}{n+1}\right)} . \end{aligned} \tag{2.17}$$

The appearance of the colour parameter λ in the above representation should be noted. Now collecting different constituting pieces (2.13)–(2.17) as

$$L_{12}^{+}(\lambda) = G_{12}^{-1}(\lambda) F_{12}^{-1} L_{12}^{0+} F_{12}^{-1} G_{12}^{-1}(\lambda)$$

and using the property $E_{ij}E_{kl} = \delta_{jk}E_{ik}$ we finally obtain the general form of $L^{\pm}$-matrices corresponding to $U_q(gl(n+1))$ as

$$L_{12}^{+}(\lambda) = \sum_{k=1}^{n+1}(\omega_k W_k^2(\lambda) T_k^2) E_{kk} + \Lambda \sum_{i<j}(\omega_i W_i(\lambda) T_i)(E_{ij+1}\hat{e}_{ji})(W_{j+1}(\lambda) T_{j+1}) \tag{2.18}$$

and similarly for

$$\begin{aligned} L^{-}_{12}(\lambda) &= G^{-1}_{12}(\lambda)F^{-1}_{12}L^{0-}_{12}F^{-1}_{12}G^{-1}_{12}(\lambda) \\ &= \sum_{k=1}^{n+1}(\omega_k^{-1}W_k^2(\lambda)T_k^2)E_{kk} - \Lambda\sum_{i<j}(W_{j+1}(\lambda)T_{j+1})(E_{j+1i}\hat{e}_{ij})(\omega_i^{-1}W_i(\lambda)T_i) \end{aligned} \tag{2.19}$$

where we have set $\hat{e}_{ij} = \tilde{e}_{ij}q^{(j-i+1)}$ Note again that due to the property $\sigma\mathcal{F}^{-1}_{12} = \mathcal{F}^{-1}_{21} = \mathcal{F}_{12}$ and similarly $\sigma\mathcal{G}^{-1}_{12} = \mathcal{G}^{-1}_{21} = \mathcal{G}_{12}$ the twisting parts are the same for both L^{+}_{12} and L^{-}_{12}.

Next we intend to construct a general colour BGR solution related to the fundamental representation of $gl(n+1)$ through the irrep $(\pi_\lambda \otimes \pi_\mu)\mathcal{R}$ of the universal $\mathcal{R}$-matrix (2.1) and (2.6). Note that we have already obtained a 'one-space' representation (2.18) and (2.19) and therefore we are going to use this result to obtain the required 'two-space' representation. Again we start from the core solution $\mathcal{R}^{0+}_{12} = L^{0+}_{12}$ and reduce it further to

$$\begin{aligned} R^{0+}_{12} &= (I\otimes\pi)L^{0+}_{12} \\ &= q^{-\frac{1}{n+1}}\left(\sum_{k=1}^{n+1} qE_{kk}\otimes E_{kk} + \sum_{k\neq j}^{n+1} E_{kk}\otimes E_{jj} + \Lambda\sum_{i<j}E_{ij}\otimes E_{ji}\right) \end{aligned} \tag{2.20}$$

and similarly

$$\begin{aligned} R^{0-}_{12} &= (I\otimes\pi)L^{0-}_{12} \\ &= q^{\frac{1}{n+1}}\left(\sum_{k=1}^{n+1} q^{-1}E_{kk}\otimes E_{kk} + \sum_{k\neq j}^{n+1} E_{kk}\otimes E_{jj} - \Lambda\sum_{i<j}E_{ji}\otimes E_{ij}\right) \end{aligned} \tag{2.21}$$

which correspond to the BGR related to $sl(n+1)$ [13]. Following similar reasoning and using (2.10) the twisting operator can also be reduced as

$$\begin{aligned} F^{-1}_{12} &= (\mathbf{I}\otimes\pi)F^{-1}_{12} \\ &= \sum_{k>j}^{n+1}(\Phi_{kj}^{\frac{1}{2}}E_{jj}\otimes E_{kk} + \Phi_{kj}^{-\frac{1}{2}}E_{kk}\otimes E_{jj}) \end{aligned} \tag{2.22}$$

with

$$\Phi_{kj}^{\frac{1}{2}} = \mathrm{e}^{-(\phi_{kj}-\phi_{kj-1}-\phi_{k-1j}+\phi_{k-1j-1})}$$

where the obvious property $\Phi_{kj} = \Phi^{-1}_{jk}$ should be noticed. The operator $G_{12}(\lambda)$ under such a mapping, however, exhibits a more interesting form with two colour parameters λ and μ yielding

$$\begin{aligned} G^{-1}_{12}(\lambda,\mu) &= (\mathbf{I}\otimes\pi_\mu)G^{-1}_{12}(\lambda) \\ &= \sum_{k,j=1}^{n+1} g_{kj}(\lambda,\mu)E_{kk}\otimes E_{jj} \end{aligned} \tag{2.23}$$

where

$$g_{kl}(\lambda,\mu) = \mathrm{e}^{\frac{1}{2}\left(-(\lambda-\mu)\sum_{j=1}^{n}\frac{jv_j}{n+1}+\lambda\sum_{t=l}^{n}v_t-\mu\sum_{s=k}^{n}v_s\right)} \tag{2.24}$$

with the property $g_{kl}(\lambda, \mu) = g_{lk}^{-1}(\mu, \lambda)$. Again collecting different pieces (2.20)–(2.24) we finally construct the coloured BGR (CBGR) as

$$
\begin{aligned}
R_{12}^{\pm}(\lambda, \mu) &= G_{12}^{-1}(\lambda, \mu) F_{12}^{-1} R_{12}^{0\pm} F_{12}^{-1} G_{12}^{-1}(\lambda, \mu) \\
&= q^{\mp\frac{1}{n+1}} \left(\sum_{k=1}^{n+1} g_{kk}^2 q^{\pm 1} E_{kk} \otimes E_{kk} + \sum_{k>j}^{n+1} (\Phi_{kj} g_{kj}^2 E_{jj} \otimes E_{kk} \right. \\
&\qquad \left. + \Phi_{kj}^{-1} g_{jk}^2 E_{kk} \otimes E_{jj}) \pm \Lambda \sum_{i<j(j<i)} g_{ij} g_{ji} E_{ij} \otimes E_{ji} \right).
\end{aligned}
\tag{2.25}
$$

It is remakable that the colour parameters introduced by (2.24) are actually in the factorized form as

$$
g_{kl}(\lambda, \mu) = f_k(\mu) f_l^{-1}(\lambda) \qquad f_l(\lambda) = \mathrm{e}^{\frac{\lambda}{2}\left(\sum_{j=1}^{n} \frac{j v_j}{n+1} - \sum_{t=l}^{n} v_t\right)}
\tag{2.26}
$$

which induce naturally in CBGR (2.25) a structure factorized in colour degrees of freedom. Note that this fact was conjectured in [16], which also helped to obtain a CBGR solution. Here we are able to substantiate the appearance of such factorized form as a consequence of the colour irrep of the universal $\mathcal{R}$-matrix. Rewriting (2.25) in the explicit factorized form

$$
\begin{aligned}
R_{12}^{\pm}(\lambda, \mu) &= q^{\mp\frac{1}{n+1}} \left(\sum_{k=1}^{n+1} q^{\pm 1} \frac{f_k^2(\mu)}{f_k^2(\lambda)} E_{kk} \otimes E_{kk} + \sum_{k>j}^{n+1} \Phi_{kj} \frac{f_k^2(\mu)}{f_j^2(\lambda)} E_{jj} \otimes E_{kk} \right. \\
&\qquad \left. + \Phi_{kj}^{-1} \frac{f_j^2(\mu)}{f_k^2(\lambda)} E_{kk} \otimes E_{jj} \pm \Lambda \sum_{i<j(j<i)} \frac{f_i(\mu)}{f_j(\lambda)} \frac{f_j(\mu)}{f_i(\lambda)} E_{ij} \otimes E_{ji} \right)
\end{aligned}
\tag{2.27}
$$

we notice that it coincides (apart from a normalization factor) with the solution found in [16] for the choice of their arbitrary function as $u_i^{(1)}(\lambda) = u_i^{(2)}(\lambda) = f_i^{-1}(\lambda)$. Observe that due to $\phi_{0j} = \phi_{n+1j} = 0$, the twisting factors Φ_{ij} can appear only starting from $gl(3)$, while the colour factors g_{ij} may be non-trivial for any value of n.

Having constructed the CBGR (2.25) and the related $L^{\pm}$ (2.18) and (2.19) for $gl(n+1)$ we can now formulate the corresponding colour FRT algebra given by the relations

$$
R_{12}^{+}(\lambda, \mu) L_1^{\pm}(\lambda) L_2^{\pm}(\mu) = L_2^{\pm}(\mu) L_1^{\pm}(\lambda) R_{12}^{+}(\lambda, \mu) \tag{2.28}
$$

$$
R_{12}^{-}(\lambda, \mu) L_1^{\pm}(\lambda) L_2^{\pm}(\mu) = L_2^{\pm}(\mu) L_1^{\pm}(\lambda) R_{12}^{-}(\lambda, \mu) \tag{2.29}
$$

$$
R_{12}^{+}(\lambda, \mu) L_1^{+}(\lambda) L_2^{-}(\mu) = L_2^{-}(\mu) L_1^{+}(\lambda) R_{12}^{+}(\lambda, \mu) \tag{2.30}
$$

$$
R_{12}^{-}(\lambda, \mu) L_1^{-}(\lambda) L_2^{+}(\mu) = L_2^{+}(\mu) L_1^{-}(\lambda) R_{12}^{-}(\lambda, \mu) \tag{2.31}
$$

where $L_1 = L_{13} \otimes I, L_2 = I \otimes L_{23}$. These relations may be obtained by taking colour representation $(\pi_\lambda \otimes \pi_\mu \otimes \mathbb{I})$ of the universal Yang–Baxter equations satisfied by $\mathcal{R}^{\pm}$ [17]. It may be checked that the CFRT algebra (2.28)–(2.31) using the explicit forms (2.18) and (2.19) and (2.25) gives back the universally deformed algebra (1.1) and (1.2) related to $gl(n+1)$.

3. Yang–Baxterization and construction of quantum integrable models

Finally, here we aim to construct a generalized quantum integrable system related to the universal deformation of the enveloping algebra of $gl(n+1)$. This requires the construction of spectral-parameter-dependent Lax operators of the models as well as the corresponding quantum R-matrix depending again on spectral parameters, such that they would satisfy the quantum Yang–Baxter equation [18]. We intend to build such operators by using the coloured objects $L^{\pm}(\lambda)$ and $R^{\pm}(\lambda, \mu)$ obtained in the previous section as building blocks and introducing the spectral parameters by Yang–Baxterization of the CFRT algebra in the form [19, 20]

$$R_{12}\left(\frac{\xi}{\eta}, \lambda, \mu\right) = \frac{\xi}{\eta} R_{12}^{+}(\lambda, \mu) - \frac{\eta}{\xi} R_{12}^{-}(\lambda, \mu) \tag{3.1}$$

for the quantum R-matrix and

$$L(\xi, \lambda) = \xi L_{12}^{+}(\lambda) + \xi^{-1} L_{12}^{-}(\lambda) \tag{3.2}$$

for the Lax operator. Here ξ and η are spectral parameters and $R_{12}^{\pm}(\lambda, \mu)$ and $L_{12}^{\pm}(\lambda)$ are the solutions of the CFRT algebra (2.28)–(2.31). Note that such a programme was carried out in [15] for the particular case $gl(2)$. Now it can be shown that for (3.1) and (3.2) to become a solution of the quantum Yang–Baxter equation

$$R_{12}\left(\frac{\xi}{\eta}, \lambda, \mu\right) L_1(\xi, \lambda) L_2(\eta, \mu) = L_2(\eta, \mu) L_1(\xi, \lambda) R_{12}\left(\frac{\xi}{\eta}, \lambda, \mu\right) \tag{3.3}$$

where $R_{12}^{\pm}(\lambda, \mu)$ and $L_{12}^{\pm}(\lambda)$ apart from being solutions of the CFRT algebra must also satisfy the relation

$$P_{12}(\lambda, \mu)[L_1^{-}(\lambda) L_2^{+}(\mu) + L_1^{+}(\lambda) L_2^{-}(\mu)] = [L_2^{+}(\mu) L_1^{-}(\lambda) + L_2^{-}(\mu) L_1^{+}(\lambda)] P_{12}(\lambda, \mu) \tag{3.4}$$

where

$$R_{12}^{+}(\lambda, \mu) - R_{12}^{-}(\lambda, \mu) = P_{12}(\lambda, \mu) = G_{12}^{-1}(\lambda, \mu) P_{12} G_{12}^{-1}(\lambda, \mu). \tag{3.5}$$

Here the operator $P_{12}(\lambda, \mu)$ may be considered as the coloured permutation operator, since P_{12} exhibits the permutation property $P_{12}(a \otimes b) = (b \otimes a) P_{12}$, whereas the relation (3.5) itself is like the coloured Hecke condition. However, while in the colourless case the relation (3.4) is trivial due to the permutation property of P_{12}, it requires to be established in our coloured case. Fortunately, we are able to prove the following lemma for $gl(n+1)$, which in turn readily shows the validity of (3.4) in this case.

Lemma. In the general $gl(n+1)$ case the coloured permutation operator exhibits the property of permutation with change of colour as

$$P_{12}(\lambda, \mu) L_1^{\mp}(\lambda) L_2^{\pm}(\mu) = L_2^{\mp}(\mu) L_1^{\pm}(\lambda) P_{12}(\lambda, \mu). \tag{3.6}$$

To prove these equalities we use the explicit forms of $L_{13}^{\pm}(\lambda)$ as found in (2.18) and (2.19) and the structure of $P_{12}(\lambda, \mu)$ as in (3.5), remembering the permutation property of P_{12} valid in the colour-free case. This gives, in particular, for the form

$$L_{13}^{-}(\lambda) = G_{13}^{-1}(\lambda)(L_{13}^{-}) G_{13}^{-1}(\lambda)$$

the relation

$$P_{12}(\lambda,\mu)L_1^-(\lambda)=\tilde{L}_2^-(\lambda,\mu)P_{12}(\lambda,\mu) \tag{3.7}$$

where $\tilde{L}_2^-(\lambda,\mu)=G_{23}^{-1}(\lambda)(\tilde{L}_{23}^-(\lambda,\mu))G_{23}^{-1}(\lambda)$ and $\tilde{L}_{23}^-(\lambda,\mu)$ differes from L_{23}^- by only a factor $J(\lambda,\mu)\equiv e^{\frac{\mu-\lambda}{2}\sum_{l=i}^{j}v_l}$ before the term $E_{j+1i}\hat{e}_{ij}$.

A similar relation holds also for $L_2^+(\mu)$, where $\tilde{L}_1^+(\lambda,\mu)=G_{13}^{-1}(\mu)(\tilde{L}_{13}^+(\lambda,\mu))G_{13}^{-1}(\mu)$ and $\tilde{L}_{13}^+(\lambda,\mu)$ may be obtained from L_1^+ with the appearance of an extra factor $J^{-1}(\lambda,\mu)$. However, again using the interesting property of interchanging colours as

$$W_k(\lambda)\hat{e}_{ji(ij)}W_l(\mu)=J^{\mp1}(\lambda,\mu)W_k(\mu)\hat{e}_{ji(ij)}W_l(\lambda) \tag{3.8}$$

one can cancel the extra factors with $J^{\pm1}(\lambda,\mu)$ appearing in $\tilde{L}_i^{\mp}(\lambda,\mu)$ as well as interchange the colours recovering

$$\tilde{L}_2^-(\lambda,\mu)\tilde{L}_1^+(\lambda,\mu)=L_2^-(\mu)L_1^+(\lambda)\,. \tag{3.9}$$

Coupling the relations like (3.7) with (3.9) we finally obtain

$$P_{12}(\lambda,\mu)L_1^-(\lambda)L_2^+(\mu)=L_2^-(\mu)L_1^+(\lambda)P_{12}(\lambda,\mu)\,. \tag{3.10}$$

The other relation of the lemma is similarly proven.

Therefore the Yang–Baxterization (3.1) and (3.2) goes through for the coloured FRT algebra solutions (2.18) and (2.19) and (2.25) related to the universal deformation of $gl(n+1)$. As a consequence the spectral-parameter-dependent quantum R-matrix (3.1) is given in the explicit form finally as

$$\begin{aligned}R\left(\frac{\xi}{\eta},\lambda,\mu\right)&=\sum_{k=1}^{n+1}g_{kk}^2(\lambda,\mu)\sin(\tilde{\lambda}-\tilde{\mu}+\alpha)E_{kk}\otimes E_{kk}\\&\quad+\sum_{k>j}^{n+1}\sin(\tilde{\lambda}-\tilde{\mu})\big(\Phi_{kj}g_{kj}^2(\lambda,\mu)E_{jj}\otimes E_{kk}+\Phi_{kj}^{-1}g_{jk}^2(\lambda,\mu)E_{kk}\otimes E_{jj}\big)\\&\quad+\sin\alpha\sum_{i<j}g_{ij}(\lambda,\mu)g_{ji}(\lambda,\mu)(E_{ij}\otimes E_{ji}+E_{ji}\otimes E_{ij})\end{aligned} \tag{3.11}$$

where the matrix is normalized by a constant and the notations

$$\xi=e^{i\tilde{\lambda}}\qquad\eta=e^{i\tilde{\mu}}\qquad q=e^{i\alpha}$$

have been introduced. It is remarkable that interpreting the colour parameters λ,μ also as the spectral parameters one gets from (3.11) an R-matrix with two sets of spectral parameters, while considering further the choice $\lambda=\tilde{\lambda}$ and $\mu=\tilde{\mu}$, equation (3.11) becomes a quantum R-matrix with non-additive dependence on spectral parameters. Recall that in standard quantum integrable models [21, 22] the known quantum R-matrices exhibit dependence on spectral parameters only additively, i.e. as $R(\lambda-\mu)$. Therefore the non-additive dependence obtained here, which for $gl(2)$ shows dependence on $(\lambda-\mu)$ and $(\lambda+\mu)$ only [15] might be of significant interest in the context of quantum integrable systems. On the other hand the corresponding Lax operator derived from (3.2) has the form

$$L(\xi,\lambda)=\sum_{k=1}^{n+1}(\xi\omega_k+\xi^{-1}\omega_k^{-1})W_k^2(\lambda)T_k^2E_{kk}$$
$$+\Lambda\sum_{i<j}\{\xi(\omega_iW_i(\lambda)T_i)(E_{ij+1}\hat{e}_{ji})(W_{j+1}(\lambda)T_{j+1})$$
$$-\xi^{-1}(W_{j+1}(\lambda)T_{j+1})(E_{j+1i}\hat{e}_{ij})(\omega_i^{-1}W_i(\lambda)T_i)\}. \tag{3.12}$$

Note that again considering the colour parameters as the spectral parameters and choosing the degenerate case $\lambda=\tilde{\lambda}$ we can get a $(n+1)\times(n+1)$-Lax operator of a generalized integrable model. Though the structure of (3.12) is apparently complicated, its construction clearly shows that it has a factorized form

$$L(\xi,\lambda)=G_{12}^{-1}(\lambda)F_{12}^{-1}(\phi_{ij})L(\xi)G_{12}^{-1}(\lambda)F_{12}^{-1}(\phi_{ij})$$

where $L(\xi)$ may be related to the Toda field models [23]. Therefore the integrable model represented by (3.11) and (3.12) may be considered as a coloured as well as twisted generalization of the exact lattice version of the Toda field model. This and other applications to integrable models are shown in the next section.

4. Application to integrable models

Let us show first that our construction of the quantum R-matrix (3.11), Lax operator (3.12) and also universal $\mathcal{R}$-matrix (2.1) recover many earlier results at different particular limits. For example, at $n=1$ and in the spectral-parameter-independent case the universal $\mathcal{R}$-matrix (2.1), as reduced to (2.1′), recovers the $\mathcal{R}$-matrix of Burdik *et al* [14] for $v_1=\mathrm{i}\alpha$, while for v_1 as an arbitrary parameter we get the recent construction of Jagannathan *et al* [12]. The spectral-parameter-dependent R-matrix (3.11), on the other hand, can be considered as a coloured generalization of the Perk–Schultz model [24]. We observe that at $f_k(\lambda)$ independent of the colour parameters and with the special choice $f_k(\lambda)=1$ and $f_k(\mu)=X_k$ one recovers the Perk–Schultz model

$$R(u)=\sum_{k=1}^{n+1}X_k^2\sin(u+\alpha)E_{kk}\otimes E_{kk}+\sum_{k>j}^{n+1}\sin u\big(\Phi_{kj}X_k^2E_{jj}\otimes E_{kk}+\Phi_{kj}^{-1}X_j^2E_{kk}\otimes E_{jj}\big)$$
$$+\sin\alpha\sum_{i<j}X_iX_j(E_{ij}\otimes E_{ji}+E_{ji}\otimes E_{ij}) \tag{4.1}$$

thus showing an unexpected connection between an integrable statistical model, constructed with the demand of a 'particle conserving' ice rule and our matrix derived from the universal $\mathcal{R}$-matrix of reductive Lie algebra constructed through twisting transformation. The additional parameters $\epsilon_\alpha=\pm1$ appearing in the diagonal elements of the Perk–Scshultz model can be generated if one starts from a universal $\mathcal{R}$-matrix related to non-compact groups. In analogy with the exact solution of the multicomponent six-vertex model [25] based on [24] it would also be interesting to now solve the more general integrable model (3.11) with dependence on the colour parameters.

We note further that the R-matrix solution for arbitrary n obtained in [15, 16] by a symmetry transformation and the (2×2)-L operator constructed through some ansatz also follow from our systematic construction starting from the universal $\mathcal{R}$-matrix of $U_q(gl(n+1))$. In [15] the L operator was found for the $n = 1$ case and the R-matrix constructed was conjectured to be obtainable from the universal $\mathcal{R}$-matrix for arbitrary n. This conjecture is proved through our construction and, moreover, we find the colour-generalized Lax operator (3.12) for general n obtained again as a representation of the universal $\mathcal{R}$-matrix. Our construction also explains and justifies some other results of an algebraic nature, the occurance of which appeared to be rather mysterious in [15].

For possible applications of our construction to integrable systems let us first limit ourselves to the fundamental representation and consider the quantum chain model with nearest-neighbour interaction. Note that in the untwisted case, i.e. for $v_j = 0$, $\phi_{ij} = 0$ and for the fundamental models we have $L^0_{am}(\xi = 1) = P_{am}$, where P_{am} is the permutation operator. Therefore one can construct the Hamiltonian H_0 of the corresponding integrable quantum model through a usual procedure [21] as

$$H_0 = \tau'\tau^{-1} \qquad \text{where} \qquad \tau = \operatorname{tr} T(\xi = 1) \qquad \tau' = \frac{\partial}{\partial\xi}(\operatorname{tr} T(\xi = 1)) .$$

Here $T(\xi) = L_N(\xi)L_{N-1}(\xi)\ldots L_1(\xi)$ is the monodromy matrix of the related integrable model. This generates

$$H_0 = \sum_m (PL^{0'})_{m,m+1}$$

or in the explicit form

$$\begin{aligned} H_0 = \sum_m (H_0)_{m,m+1} = \sin^{-1}\alpha \sum_m \Bigg(\tfrac{1}{2}\cos\alpha \sum_{k,j}^{n+1} \big(E^{(m)}_{kk} \otimes E^{(m+1)}_{kk} - E^{(m)}_{jj} \otimes E^{(m+1)}_{kk} \\ - E^{(m)}_{kk} \otimes E^{(m+1)}_{jj} \big) + \sum_{k \neq l} \big(E^{(m)}_{kl} \otimes E^{(m+1)}_{lk} + E^{(m)}_{lk} \otimes E^{(m+1)}_{kl} \big) \Bigg) \end{aligned} \tag{4.2}$$

where some constant terms are ignored. Here and in what follows the superscript (m) denotes the mth lattice site. Now for finding the extension of the above system with non-trivial twisting parameters v_j and ϕ_{ij} preserving the integrability, we observe that in this general case one gets

$$L_{1m}(\xi = 1) = G^{-1}_{1m} F^{-1}_{1m} P_{1m} F^{-1}_{1m} G^{-1}_{1m} = (A_1 A^{-1}_m)^{(Z^{(1)} - Z^{(m)})} P_{1m}$$

where $A_m = e^{\frac{1}{2}\sum_{k,j} v_j a^{jk} H^{(m)}_k}$. Now we may consider two different situations when

(i) the colour parameter $Z^{(1)}$ is independent of the spectral parameter or
(ii) $Z^{(1)}$ is a function of spectral parameter λ'.

In both cases the above described construction goes through almost parallely and in the first case we obtain the new Hamiltonian

$$H = \sum_m (H)_{m,m+1} = \sum_m g^{-1}_{m,m+1} f^{-1}_{m,m+1} (H_0)_{m,m+1} f_{m,m+1} g_{m,m+1} \tag{4.3}$$

where

$$g_{m,m+1} = \mathrm{e}^{-Z^{(1)} \sum_{j,l} v_j a^{jl} (H_l^{(m)} + H_l^{(m+1)}) + \frac{1}{2}(Z^{(m)} + Z^{(m+1)}) \sum_{j,l} v_j a^{jl} (H_l^{(m+1)})}$$

and

$$f_{m,m+1} = \mathrm{e}^{\sum_{j,l} \phi_{jl} H_l^{(m)} H_j^{(m+1)}} .$$

The explicit form of this Hamiltonian may be calculated from (4.3) after some algebra as

$$\begin{aligned} H &= \sum_m (H)_{m,m+1} \\ &= \sin^{-1}\alpha \sum_m \Bigg(\tfrac{1}{2}\cos\alpha \sum_{k,j}^{n+1} \left(E_{kk}^{(m)} E_{kk}^{(m+1)} - E_{jj}^{(m)} E_{kk}^{(m+1)} - E_{kk}^{(m)} E_{jj}^{(m+1)} \right) \\ &\qquad + \sum_{k<l} \left(\tilde{f}_{m,m+1}^{-1} E_{kl}^{(m)} E_{lk}^{(m+1)} + \tilde{f}_{m,m+1} E_{lk}^{(m)} E_{kl}^{(m+1)} \right) \Bigg) \end{aligned} \tag{4.4}$$

where

$$\tilde{f}_{m,m+1} = \mathrm{e}^{\frac{1}{2} \sum_{s=k}^{l-1} ((Z^{(m)} + Z^{(m+1)}) v_s + \sum_{j=1}^{n} a_{js} (\phi_{jk} + \phi_{jl} - \phi_{jk-1} - \phi_{jl-1}))} .$$

To see the physical significance of such extensions we look into the structure of the Hamiltonian (4.4) for the particular case $n = 1$, when there is only one parameter v_1 and no ϕ_{kl}. Introducing the Pauli matrices we get

$$\begin{aligned} H &= \sum_m (H)_{m,m+1} \\ &= \sin^{-1}\alpha \sum_m \Big(\tfrac{1}{2}\cos\alpha\, \sigma_3^{(m)} \sigma_3^{(m+1)} + \mathrm{e}^{\frac{1}{2}(Z^{(m)} + Z^{(m+1)}) v_1} \sigma_+^{(m)} \sigma_-^{(m+1)} \\ &\qquad + \mathrm{e}^{-\frac{1}{2}(Z^{(m)} + Z^{(m+1)}) v_1} \sigma_-^{(m)} \sigma_+^{(m+1)} \Big) . \end{aligned} \tag{4.5}$$

Note that for $Z^{(m)} = 1$ the model (4.5) becomes the Wu–Mackoy model [26] of a ferroelectric in a constant external electric field v_1. Therefore (4.5) is a generalization of the Wu–Mackoy model with the inclusion of inhomogenious electric field $Z^{(m)} v_1$. Its exact solution through the Bethe ansatz would be a tractable and physically important problem.

If we now consider the colour parameter $\dot{Z}^{(m)}$ to be dependent on the spectral parameters, instead of (4.3) we get

$$H = \sum_m \left(g_{m,m+1}^{-1} f_{m,m+1}^{-1} (H_0)_{m,m+1} f_{m,m+1} g_{m,m+1} - Z^{(1)'}(0) \sum_{j,l} v_j a^{jl} H_l^{(m)} \right) . \tag{4.6}$$

For $n = 1$ the additional term in the Hamiltonian (4.6) becomes $Z^{(1)'}(0) v_1 \sigma_3^{(m)}$ describing the interaction of the extended inhomogeneous Wu–Mackoy model with an external magnetic field.

Let us now focus on non-fundamental integrable quantum models, which may be generated by the Lax operator (3.12) and rewrite it in the form

$$L(\xi, \lambda) = \sum_{k=1}^{n+1} (\xi \tau_k^+ + \xi^{-1} \tau_k^-) E_{kk} + \Lambda \sum_{i<j} \xi (E_{ij+1} \tau_{ji}) - \xi^{-1} (E_{j+1i} \tau_{ij}) \tag{4.7}$$

where

$$\tau_k^{\pm} = (\omega_k)^{\pm 1}(W_k(\lambda)T_k)^2 \tag{4.8}$$

$$\tau_{ji} = (W_i(\lambda)T_i)(\omega_i \hat{e}_{ji})(W_{j+1}(\lambda)T_{j+1}) \tag{4.9}$$

and

$$\tau_{ij} = (W_{j+1}(\lambda)T_{j+1})(\hat{e}_{ij}\omega_i^{-1})(W_i(\lambda)T_i) \qquad i < j \tag{4.10}$$

with

$$\omega_k = q^{\left(\sum_{s=k}^n H_s - \sum_{i=1}^n \frac{iH_i}{n+1}\right)}$$

for $k = 1, \ldots, n$, while $\omega_{n+1} = q^{-\sum_{i=1}^n \frac{iH_i}{n+1}}$ and

$$T_k = \mathrm{e}^{-\sum_{j=1}^n (\phi_{kj} - \phi_{k-1j})H_j} \qquad W_k(\lambda) = \mathrm{e}^{\frac{1}{2}\left(\lambda \sum_{jl}^n v_j a^{jl} H_l + \mathbf{Z}(\sum_{j=1}^n \frac{jv_j}{n+1} - \sum_{s=k}^n v_s)\right)}$$

for $k = 1, \ldots, n$, while

$$W_{n+1}(\lambda) = \mathrm{e}^{\frac{1}{2}\left(\lambda \sum_{jl}^n v_j a^{jl} H_l + \mathbf{Z} \sum_{j=1}^n \frac{jv_j}{n+1}\right)}.$$

It is interesting to note that realizations of the underlying quantized algebra in different physical variables, e.g. bosonic, q-bosonic [27–29], or canonical variables would result, in principle, in the generation of different kinds of lattice models from (4.7), which would be exactly integrable at the quantum level. To demonstrate this we first use the following q-bosonic realization [30] of the quantized algebra:

$$H_i = N_i - N_{i+1} \qquad N_1 = (n+1)s - \sum_{k=2}^{n+1} N_k \qquad \text{for} \quad i = 1, 2 \ldots, n \tag{4.11}$$

and

$$X_k^+ = A_k^\dagger A_{k+1} \qquad X_1^+ = ([N_1]_q)^{\frac{1}{2}} A_2 \qquad X_k^- = (X_k^+)^\dagger \tag{4.12}$$

with $k = 2, 3 \ldots, n$, and $[x]_q = \frac{\sin \alpha x}{\sin \alpha}$. Here s is an arbitrary parameter and $A_k^\dagger, A_k$ are n number of q-bosonic operators satisfying the deformed commutation relation

$$[A_k, A_l^\dagger] = \delta_{kl} \frac{\cos(\alpha(N_k + \frac{1}{2}))}{\cos\frac{\alpha}{2}} \qquad [N_k, A_l^\dagger] = \delta_{kl} A_l^\dagger \qquad [N_k, A_l] = -\delta_{kl} A_l . \tag{4.13}$$

In this case (4.7) may be considered as the representative Lax operator of a novel quantum integrable multimode q-bosonic model. Indeed, one observes that (4.11) transforms the sums as

$$\sum_{s=k}^n H_s = N_k - N_{n+1} \qquad \sum_{i=1}^n \frac{iH_i}{n+1} = s - N_{n+1}$$

giving $\omega_k = q^{N_k - s}$ for $k = 1, 2, \ldots, n$ and $\omega_{n+1} = q^{N_{n+1} - s}$. Consequently, inserting the realization of quantum algebra (4.11) and (4.12) in (4.8)–(4.10) and taking into account the relation (1.13c) between $\hat{e}_{ij}$ and the generators $X_k^{\pm}$, after some algebra one arrives at (4.7) containing

$$\tau_i^{\pm} = q^{\pm(N_i - s)}(W_i T_i)^2 \tag{4.14}$$

$$\tau_{j1} = W_1 T_1 (A_j^{\dagger} q^{-\frac{m_j}{2}} [N_1]_q^{\frac{1}{2}}) W_{j+1} T_{j+1} \tag{4.15}$$

$$\tau_{1j} = W_{j+1} T_{j+1} ([N_1]_q^{\frac{1}{2}} q^{\frac{m_j}{2}} A_j) W_1 T_1 \tag{4.16}$$

$$\tau_{ij} = W_{j+1} T_{j+1} (A_i^{\dagger} q^{-\frac{1}{2}(m_i - m_j)} A_j) W_i T_i \qquad \text{for} \quad i < j \qquad \text{and} \qquad \tau_{ji} = (\tau_{ij})^{\dagger} \tag{4.17}$$

with $i, j = 2, 3, \ldots, n+1$ and $m_j = \sum_{r=2, r \neq j} \mathrm{sgn}(j - r) N_r$, $m_1 = 0$. Using the isomorphism of the algebra some irrelevant factors are absorbed in the above generators to make the realization simpler.

Here T_k and W_k operators take the form

$$T_k = \mathrm{e}^{-\sum_{j=1}^{n} (\phi_{kj} - \phi_{k-1j})(N_j - N_{j+1})} \tag{4.18}$$

and

$$W_k = \mathrm{e}^{\frac{1}{2}\left(\lambda \sum_{jl}^{n} v_j a^{jl} (N_l - N_{l+1}) + \mathrm{Z}(\sum_{j=1}^{n} \frac{j v_j}{n+1} - \sum_{s=k}^{n} v_s)\right)}. \tag{4.19}$$

Note that a similar representation through q-bosons in the case when $T_k = 0 = W_k$ was also found in [20]. The present model apart from the q-bosonic creation, annihilation and the number operators also contains an 'external' operator Z and a set of parameters λ, v_j and ϕ_{ij}. This is an exactly integrable q-bosonic quantum lattice model.

Between the usual bosons having the standard commutation relations

$$[a_k, a_l^{\dagger}] = \delta_{kl} \qquad [N_k, a_l^{\dagger}] = \delta_{kl} a_l^{\dagger} \qquad [N_k, a_l] = -\delta_{kl} a_l$$

and the q-bosons there exists a simple mapping

$$A_j = a_j \left(\frac{[N_j]_q}{N_j}\right)^{\frac{1}{2}} \qquad A_j^{\dagger} = \left(\frac{[N_j]_q}{N_j}\right)^{\frac{1}{2}} a_j^{\dagger}$$

which helps to also realize the Lax operator of the q-bosonic model in standard bosons.

Another interesting model which generalizes the lattice regularized version of the Toda field model [23] in a non-trivial way can be constructed again starting from (4.7). For this using the results of [23] we can find a realization of q-bosons in canonical variables $[u_a^{(m)}, p_b^{(n)}] = \mathrm{i}\delta_{nm}\delta_{ab}$ given by

$$A_i^{(m)} = q^{(\vec{\rho}_i \ \vec{p}^{(m)})} [N_i]_q \qquad A_i^{(m)\dagger} = [N_i]_q \ q^{-(\vec{\rho}_i \ \vec{p}^{(m)})} \qquad N_k^{(m)} = \vec{\alpha}_k \ \vec{u}^{(m)} + s \tag{4.20}$$

with $i = 2, 3, \ldots, n+1$ and $k = 1, 2, \ldots, n+1$, where $\vec{\alpha}_i$ and $\vec{\rho}_i$ are the simple roots and fundamental weights of $sl(n+1)$, respectively, with the relation

$$(\vec{\alpha}_i \cdot \vec{\rho}_j) = \delta_{ij} \qquad i, j = 2, 3, \ldots, n+1\,.$$

Here we have assumed $\vec{\alpha}_1 = -\sum_{i=2}^{n+1} \vec{\alpha}_i$ and $\vec{\rho}_1 = 0$. Now inserting (4.20) in (4.14)–(4.19) one obtains the Lax operator in the form (4.7), representing a generalization of the well known [23] quantum integrable lattice Toda field model. By rescaling $\vec{p} \to \Delta\vec{p}$

and adjusting the parameters one can introduce the lattice constant Δ. Note that the generalization over the known model is achieved here due to the presence of the terms T_k and W_k, which are originated in our model through twisting transformations. They enter into the Lax operator in a non-trivial way and apart from the operators N_k also contain the 'external field' $Z^{(m)}$ and the parameters v_j, λ and ϕ_{ij} with $i, j = 1, 2, \ldots, n$.

To see its explicit form we consider for simplicity the $n = 1$ case, which corresponds to a generalization of the lattice sine–Gordon model [31] through the inclusion of non-trivial v, λ , $Z^{(m)}$:

$$L^{(m)}(\lambda, \xi) = \mathrm{e}^{\mathrm{i}v\lambda u^{(m)}} \times \begin{pmatrix} \frac{\mathrm{i}m_0\Delta}{2}(\xi \mathrm{e}^{\mathrm{i}\alpha u^{(m)}} - \frac{1}{\xi}\mathrm{e}^{-\mathrm{i}\alpha u^{(m)}})\mathrm{e}^{-vZ^{(m)}} & g(u^{(m)})\mathrm{e}^{\mathrm{i}\Delta p^{(m)}} \\ \mathrm{e}^{-\mathrm{i}\Delta p^{(m)}} g(u^{(m)}) & \frac{\mathrm{i}m_0\Delta}{2}(\xi \mathrm{e}^{-\mathrm{i}\alpha u^{(m)}} - \frac{1}{\xi}\mathrm{e}^{\mathrm{i}\alpha u^{(m)}})\mathrm{e}^{vZ^{(m)}} \end{pmatrix} \tag{4.21}$$

where

$$g^2(u^{(m)}) = 1 + \tfrac{1}{2}m_0^2\Delta^2 \cos\alpha(2u^{(m)} + 1)\,.$$

It would be an interesting problem to find and investigate the field models corresponding to such generalized quantum integrable discrete systems.

5. Concluding remarks

We have constructed the universal $\mathcal{R}$-matrix intertwining between the coproduct structures related to the universal deformation of the reductive Lie algebras. Such constructions along with establishing the property of quasitriangularity of the associated Hopf algebra can also be applied for building the quantum R-matrices as well as the Lax operators of different classes of quantum integrable models. This programme is carried out exploiting the universal $\mathcal{R}$-matrix related to $gl(n+1)$. The explicit form of the quantum R-matrix and the Lax operator depend upon the spectral as well as colour parameters apart from a set of additional parameters, introduced through twisting. The integrable systems represented by such objects range from statistical models, quantum spin chains to the generalization of lattice regularized Toda field theory. This systematic construction scheme starts from the universal R-matrix and goes through several intermediate steps like formulation of the coloured Faddeev–Reshetikhin–Takhtajan (CFRT) algebra, coloured braid-group, Yang–Baxterization of CFRT etc, which may be considered as important byproducts.

Similar investigation related to other types of Lie algebras would be undoubtadely a desirable one.

Appendix. A note on the explicit form of the universal (untwisted) $\mathcal{R}$-matrix

Here we want to prove that

$$\mathcal{R} = \check{R}K = K\check{R}' \tag{A.1}$$

where the relevant quantities are defined in the text.

We first consider generators $e_{\pm\ell}$ associated to *simple roots* and calculate

$$\begin{aligned}(e_\ell \otimes 1)K &= \prod_{ij}\sum_{n\geqslant 0}\frac{(c_{ij})^n}{n!}(H_i - a_{i\ell})^n \otimes H_j^n\ (e_\ell\otimes 1)\\ &= \prod_{ij}\exp[c_{ij}(H_i\otimes H_j) - c_{ij}(1\otimes a_{i\ell}H_j)](e_\ell\otimes 1)\\ &= K\left(e_\ell \otimes q_{\ell\ell}^{-\frac{H_\ell}{2}}\right).\end{aligned} \tag{A.2}$$

In fact,

$$\begin{aligned}\sum_{ij} c_{ij}a_{i\ell}H_j &= \sum_{ij} h_{\rho(i)}\frac{(\alpha_i\cdot\alpha_i)}{2}\frac{(\alpha_j\cdot\alpha_j)}{2}2\frac{d_{\ell i}}{(\alpha_i\cdot\alpha_i)}d^{ij}H_j\\ &= \sum_j \frac{(\alpha_j\cdot\alpha_j)}{2}H_j\sum_i h_{\rho(i)}d_{\ell i}d^{ij}\\ &= \sum_j\frac{(\alpha_j\cdot\alpha_j)}{2}h_{\rho(\ell)}\delta_{j\ell}H_j = h_{\rho(\ell)}\frac{(\alpha_\ell\cdot\alpha_\ell)}{2}H_\ell\end{aligned}$$

where we used the fact that in the sum over i only terms with ℓ and i in the same simple component may give a non-vanishing contribution and in this case $h_{\rho(i)} = h_{\rho(\ell)}$ goes out of the sum leaving a $\delta_{j\ell}$.

Analogously,

$$(1\otimes e_{-\ell})K = K\left(q_{\ell\ell}^{\frac{H_\ell}{2}}\otimes e_{-\ell}\right). \tag{A.3}$$

We conclude from (2) and (3) that

$$\begin{aligned}(e_\ell\otimes e_{-\ell})\ K &= (e_\ell\otimes 1)(1\otimes e_{-\ell})\ K\\ &= K\ (e_\ell\ q_{\ell\ell}^{\frac{H_\ell}{2}}\otimes q_{\ell\ell}^{-\frac{H_\ell}{2}}\ e_{-\ell})\\ &= K\ (e_\ell\ q_\rho^{h_\ell}\otimes q_\rho^{-h_\ell}e_{-\ell})\end{aligned}$$

with $h_\ell = \frac{(\alpha_\ell\cdot\alpha_\ell)}{2}H_\ell$ as in the paper.

Suppose now e_γ is a generator associated to a *non-simple root*. Then e_γ can be expressed as a multiple q-commutator of generators associated to simple roots, hence we can write

$$e_\gamma = \sum_{i_1\ldots i_n} c_{i_1\ldots i_n}e_{i_1}\ldots e_{i_n}$$

where each term in the sum contains the same generators (but in a different order) $e_{i_1},\ldots,e_{i_n}$ associated to simple roots. Moreover, as in the main text, we can introduce the Cartan generator associated to the non-simple root $h_\gamma = \sum_{1\leqslant k\leqslant n} h_{i_k}$.

We now calculate, using (2),

$$\begin{aligned}\left(\sum_{i_1\ldots i_n} c_{i_1\ldots i_n}e_{i_1}\ldots e_{i_n}\otimes 1\right)K &= \sum_{i_1\ldots i_n} c_{i_1\ldots i_n}(e_{i_1}\otimes 1)\ldots(e_{i_n}\otimes 1)K\\ &= K\sum_{i_1\ldots i_n} c_{i_1\ldots i_n}e_{i_1}\ldots e_{i_n}\otimes q_\rho^{-h_\gamma} = K(e_\gamma\otimes q_\rho^{-h_\gamma}).\end{aligned}$$

By performing the analogue calculation for $1\otimes e_{-\gamma}$ we conclude that

$$(e_\gamma\otimes e_{-\gamma})K = K(e_\gamma q_\rho^{h_\gamma}\otimes q_\rho^{-h_\gamma}e_{-\gamma}).$$

From the latter equality the proof of (1) follows easily.

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